Four-state models and Clifford algebras

1994 J. Phys. A: Math. Gen. 27 5393

(http://iopscience.iop.org/0305-4470/27/16/008)

J. Phys. A: Math. Gen. 27 (1994) 5393–5407 Printed in the UK

Four-state models and Clifford algebras

Haye Hinrichsen†

Freie Universität Berlin, Institut für Theoretische Physik, Arnimallee 14, D-14195 Berlin, Germany

Received 18 April 1994, in final form 29 June 1994

Abstract. With appropriate boundary conditions the anisotropic XY chain in a magnetic field is known to be invariant under quantum group transformations. We generalize this model defining a class of integrable chains with several fermionic degrees of freedom per site. In order to maintain the quantum group symmetry a general condition on the parameters of these systems is derived. It is shown that the corresponding quantum algebra is a multi-parameter deformation of the Clifford algebra. Discussing a special physical example we observe a new type of zero mode.

1. Introduction

In statistical mechanics the anisotropic XY chain is one of the simplest exactly solvable models. Its L-site Hamiltonian with periodic boundary conditions

$$H^{XY}_{\mathrm{per}}(\eta, h) = -\tfrac{1}{2}\sum_{j=1}^{L}\left(\eta\sigma_j^x\sigma_{j+1}^x + \eta^{-1}\sigma_j^y\sigma_{j+1}^y\right) - h\sum_{j=1}^{L}\sigma_j^z \tag{1.1}$$

depends on two parameters, namely the anisotropy parameter η and the magnetic field h ($\sigma_j^{x,y,z}$ are Pauli matrices acting on site j). This Hamiltonian appears in the domain wall theory of two-dimensional commensurate–incommensurate phase transitions [1, 2] and provides a good model for Helium adsorbed on metallic surfaces. It also describes the master equation of the kinetic Ising model [3] and plays a role in one-dimensional reaction–diffusion processes [4].

The present work is based on the investigation of the anisotropic XY chain with a *special kind of boundary conditions* defined by the Hamiltonian

$$H^{XY}(\eta, q) = -\tfrac{1}{2}\sum_{j=1}^{L-1}\left(\eta\,\sigma_j^x\sigma_{j+1}^x + \eta^{-1}\sigma_j^y\sigma_{j+1}^y + q\,\sigma_j^z + q^{-1}\sigma_{j+1}^z\right) \tag{1.2}$$

where q is related to the magnetic field by $2h = q + q^{-1}$ (notice that compared to (1.1) there are additional surface fields at the ends of the chain). These boundary conditions make the system invariant under quantum group transformations [5–7]. Beside their mathematical relevance these boundary conditions are also of physical interest since they appear naturally in a special one-dimensional reaction–diffusion process with open ends [4].

Beside the XY chain there are many other quantum chains where a quantum group symmetry can implemented by choosing appropriate boundary conditions with q-dependent

† E-mail address: hinrichs@aster.physik.fu-berlin.de

surface terms. As an important example there is the class of $SU(P|M)_q$-invariant Perk–Schultz chains [8, 9] which includes the isotropic XY chain [5] and the XXZ Heisenberg chain [10]. Quantum group symmetries may also play a role for quantum chains in the thermodynamic limit [11] and for chains with cyclic boundary conditions, for example, the XXZ chain with toroidal boundary conditions [12] and n-state Vertex models with periodic boundary conditions [13].

The attempts to introduce diagonalizible generalizations of the XY model go back to Suzuki [14]. Following these ideas we consider generalized quantum chains with a higher number of degrees of freedom per site. We maintain the quantum group symmetry by choosing special boundary conditions and imposing appropriate restrictions on the parameters. We are interested in both the physical properties of these generalized chains (like their spectra) and the mathematical structure of the corresponding quantum algebra.

Let us briefly summarize the results of [6]. The XY chain Hamiltonian (1.2) is invariant under a two-parameter quantum Clifford algebra which is defined by the generators T^1, T^2 and the central element E with the commutation relations

$$\begin{aligned} &\{T^1, T^1\} = 2[E]_{\alpha_1} \qquad &&\{T^2, T^2\} = 2\,[E]_{\alpha_2} \\ &\{T^1, T^2\} = 0 \qquad &&[E, T^1] = [E, T^2] = 0 \end{aligned} \tag{1.3}$$

where α_1 and α_2 are deformation parameters and

$$[E]_{\alpha_\mu} = \frac{\alpha_\mu^E - \alpha_\mu^{-E}}{\alpha_\mu - \alpha_\mu^{-1}}\,. \tag{1.4}$$

The coproducts of these generators read

$$\begin{aligned} \Delta(T^1) &= \alpha_1^{E/2} \otimes T^1 + T^1 \otimes \alpha_1^{-E/2} \\ \Delta(T^2) &= \alpha_2^{E/2} \otimes T^2 \mp T^2 \otimes \alpha_2^{-E/2} \\ \Delta(E) &= E \otimes \mathbf{1} + \mathbf{1} \otimes E\,. \end{aligned} \tag{1.5}$$

For $\alpha_1 = \alpha_2 = 1$ the system undergoes a Pokrovski–Talapov phase transition [1]. Here the quantum algebra (1.3) reduces to the (classical) Clifford algebra

$$\{T^\mu, T^\nu\} = 2\,E\,\delta^{\mu\nu} \qquad [E, T^\mu] = 0 \qquad \mu, \nu = 1, 2\,. \tag{1.6}$$

Apart from the trivial one-dimensional representation the algebra (1.3) has only two-dimensional irreducible representations, in particular the fermionic representation corresponds to taking $T^1 = \sigma^x$, $T^2 = \sigma^y$, and $E = 1$.

The explicit expressions for the generators in the case of the XY chain can be obtained from the fermionic (one-site) representation by a multiple application of the coproducts (1.5). In order to do so, let us introduce local fermionic operators τ_j^1 and τ_j^2 by a Jordan–Wigner transformation

$$\tau_j^1 = \left(\prod_{i=1}^{j-1} \sigma_i^z\right)\sigma_j^x \qquad \tau_j^2 = \left(\prod_{i=1}^{j-1} \sigma_i^z\right)\sigma_j^y \tag{1.7}$$

which obey the Clifford algebra

$$\left\{\tau_i^\mu, \tau_j^\nu\right\} = 2\,\delta_{ij}\delta^{\mu\nu} \qquad i, j = 1, \ldots, L \quad \mu, \nu = 1, 2\,. \tag{1.8}$$

In terms of these operators the Hamiltonian (1.2) can be written as

$$H^{XY}(\eta, q) = \tfrac{1}{2} i \sum_{j=1}^{L-1} \left(\eta \tau_j^2 \tau_{j+1}^1 - \eta^{-1} \tau_j^1 \tau_{j+1}^2 + q \tau_j^1 \tau_j^2 + q^{-1} \tau_{j+1}^1 \tau_{j+1}^2\right). \tag{1.9}$$

The explicit expressions for the generators T^1, T^2 and E read

$$T^1 = \alpha_1^{-(L+1)/2} \sum_{j=1}^{L} \alpha_1^j \tau_j^1 \qquad T^2 = \alpha_2^{-(L+1)/2} \sum_{j=1}^{L} \alpha_2^j \tau_j^2 \qquad E = L \tag{1.10}$$

where

$$\alpha_1 = \frac{q}{\eta} \qquad \alpha_2 = q\eta \tag{1.11}$$

are the deformation parameters. Both of them are essential, i.e. it is impossible to remove one of the parameters by similarity transformation. Furthermore notice that the generator E simply counts the number of sites. Therefore if one of the deformation parameters is a root of unity, the irreducible representations of the algebra (1.3) depend on the length of the chain which requires the definition of a special thermodynamical limit in this case [6].

The generators (1.10) commute with the Hamiltonian and appear physically as a fermionic zero mode. This zero mode is present for arbitrary parameters q and η and causes all levels of the spectrum to be at least two-fold degenerated. We want to emphasize that such a zero mode cannot be observed in the case of periodic or free boundary conditions. In other words, the quantum group symmetry is directly related to the special boundary conditions in (1.2).

If both deformation parameters α_1 and α_2 coincide, we have the isotropic case $\eta = 1$. Here the total magnetization $S^z = \sum_{j=1}^{L} \sigma_j^z$ commutes with the Hamiltonian and generates an additional $U(1)$ symmetry. This allows the quantum algebra (1.3) to be enlarged by adding the commutation relations

$$[T^1, N] = 2\mathrm{i}T^2 \qquad [T^2, N] = -2\mathrm{i}T^1 \qquad [E, N] = 0 \tag{1.12}$$

and the coproduct

$$\Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N \tag{1.13}$$

where $N = \frac{1}{2}(S^z + L)$. The resulting algebra is the $U_q[SU(1/1)]$ superalgebra [5].

A first attempt to generalize the quantum group invariant XY chain has been made in [15]. Defining a $2M$-dimensional affine Clifford–Hopf algebra and using an R-matrix approach the author showed that the generalized XY chain introduced by Suzuki [14]

$$\tilde{H} = -\sum_{k=1}^{K} \sum_{j=1}^{L'} (\tilde{J}_{x,k} \sigma_j^x \sigma_{j+k}^x + \tilde{J}_{y,k} \sigma_j^y \sigma_{j+k}^y) \sigma_{j+1}^z \cdots \sigma_{j+k-1}^z + h \sum_{j=1}^{L'} \sigma_j^z \tag{1.14}$$

possesses a quantum group symmetry provided that $L' = Lm$ for some integer m and

$$\tilde{J}_{x,k} = -J_x \delta_{m,k} \qquad \tilde{J}_{y,k} = -J_y \delta_{m,k} \qquad k = 1, \ldots, K\,. \tag{1.15}$$

This case is trivial for the following reason. If one performs the transformation

$$\sigma_{mr+s}^{x,y} \to \left(\prod_{i=0}^{r-1} \prod_{j=s+1}^{m} \sigma_{mi+j}^z \right) \left(\prod_{i=r+1}^{L'/m} \prod_{j=1}^{s-1} \sigma_{mi+j}^z \right) \sigma_{mr+s}^{x,y} \tag{1.16}$$

$$\sigma_{mr+s}^z \to \sigma_{mr+s}^z \qquad r = 0, \ldots, L'/m;\ s = 1, \ldots, m$$

one obtains the Hamiltonian

$$\tilde{H}' = \sum_{r=0}^{L'/m-1} \sum_{s=1}^{m} \left(\tilde{J}_x \sigma_{mr+s}^x \sigma_{m(r+1)+s}^x + \tilde{J}_y \sigma_{mr+s}^y \sigma_{m(r+1)+s}^y \right) + h \sum_{r=0}^{L'/m-1} \sum_{s=1}^{m} \sigma_{mr+s}^z \tag{1.17}$$

which is a sum of m identical anisotropic XY chains. Since the transformation (1.16) does not change the algebra of the Pauli matrices, $\tilde{H}$ and $\tilde{H}'$ differ only by a similarity transformation. Therefore the physical properties of $\tilde{H}$ are already known. In this paper we show that it is possible to define *non-trivial* generalizations of the XY chain maintaining both the quantum group symmetry and the integrability in terms of free fermions. In contrast to [15] we start from the physical point of view generalizing the XY chain directly in its fermionic formulation (1.9). In order to implement the quantum group symmetry we then derive a general condition for the existence of zero modes. As an example we consider a four-state quantum chain defined on two commuting copies of Pauli matrices $\sigma_j^{x,y,z}$ and $\varrho_j^{x,y,z}$. Its Hamiltonian depends on ten parameters including one normalization parameter

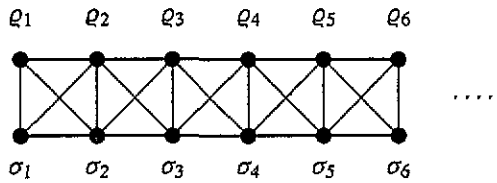
$$
\begin{aligned}
&H(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\omega_{12},\omega_{34},\omega_{14},\omega_{23},\omega_{13},\omega_{24}) \\
&\quad = -\tfrac{1}{2}\sum_{j=1}^{L-1}\big[\omega_{12}(\gamma_1^{-1}\gamma_2\sigma_j^x\varrho_j^z\sigma_{j+1}^x+\gamma_1\gamma_2^{-1}\sigma_j^y\varrho_j^z\sigma_{j+1}^y+\gamma_1\gamma_2\sigma_j^z+\gamma_1^{-1}\gamma_2^{-1}\sigma_{j+1}^z) \\
&\qquad +\omega_{34}(\gamma_3^{-1}\gamma_4\,\varrho_j^x\sigma_{j+1}^z\varrho_{j+1}^x+\gamma_3\gamma_4^{-1}\,\varrho_j^y\sigma_{j+1}^z\varrho_{j+1}^y+\gamma_3\gamma_4\,\varrho_j^z+\gamma_3^{-1}\gamma_4^{-1}\,\varrho_{j+1}^z) \\
&\qquad +\omega_{14}(\gamma_1^{-1}\gamma_4\varrho_j^x\sigma_{j+1}^x+\gamma_1\gamma_4^{-1}\sigma_j^y\varrho_j^z\sigma_{j+1}^z\varrho_{j+1}^y-\gamma_1\gamma_4\sigma_j^y\varrho_j^y-\gamma_1^{-1}\gamma_4^{-1}\sigma_{j+1}^y\varrho_{j+1}^y) \\
&\qquad -\omega_{23}(\gamma_2^{-1}\gamma_3\varrho_j^y\sigma_{j+1}^y+\gamma_2\gamma_3^{-1}\sigma_j^x\varrho_j^z\sigma_{j+1}^z\varrho_{j+1}^x-\gamma_2\gamma_3\sigma_j^x\varrho_j^x-\gamma_2^{-1}\gamma_3^{-1}\sigma_{j+1}^x\varrho_{j+1}^x) \\
&\qquad -\omega_{13}\,(\gamma_1^{-1}\gamma_3\varrho_j^y\sigma_{j+1}^x-\gamma_1\gamma_3^{-1}\sigma_j^y\varrho_j^z\sigma_{j+1}^z\varrho_{j+1}^x+\gamma_1\gamma_3\sigma_j^y\varrho_j^x+\gamma_1^{-1}\gamma_3^{-1}\sigma_{j+1}^y\varrho_{j+1}^x) \\
&\qquad +\omega_{24}(\gamma_2^{-1}\gamma_4\varrho_j^x\sigma_{j+1}^y-\gamma_2\gamma_4^{-1}\sigma_j^x\varrho_j^z\sigma_{j+1}^z\varrho_{j+1}^y+\gamma_2\gamma_4\sigma_j^x\varrho_j^y \\
&\qquad +\gamma_2^{-1}\gamma_4^{-1}\sigma_{j+1}^x\varrho_{j+1}^y)\big]\,.
\end{aligned}
\tag{1.18}
$$

Although at first sight this Hamiltonian seems to be rather artificial we will see that it is indeed a natural generalization of the XY chain Hamiltonian (1.2). We will show that this chain is invariant under a four-parameter deformation of the Clifford algebra. This quantum algebra is defined by the generators T^1, T^2, T^3, T^4 and E with the commutation relations

$$\{T^\mu, T^\nu\} = 2\delta^{\mu\nu}[E]_{\alpha_\mu} \qquad [E, T^\mu] = 0 \qquad \mu,\nu = 1,\ldots,4 \tag{1.19}$$

where $\alpha_\mu = \gamma_\mu^2$ ($\mu = 1,\ldots,4$) are four deformation parameters. As in the case of the XY chain, we observe additional symmetries if some of these parameters coincide.

The model defined in (1.18) can be understood as a system of two interacting XY chains:



In opposition to a single XY chain (which is completely described by the deformation parameters) its Hamiltonian depends on six further parameters ω_{ij} which do not occur in the quantum algebra and cannot generally be eliminated by similarity transformation. Since these parameters allow the implemention of non-trivial couplings between the XY chains without breaking the quantum group symmetry, we expect a richer structure than in the case of two decoupled XY chains as in (1.14). However, switching off these couplings by taking $\omega_{13} = \omega_{14} = \omega_{23} = \omega_{24} = 0$ and performing the following automorphism on the Pauli matrices:

$$
\begin{aligned}
&\sigma_j^{x,y} \to \left(\prod_{i=1}^{j-1}\varrho_i^z\right)\sigma_j^{x,y} \qquad \varrho_j^{x,y} \to \left(\prod_{i=j+1}^{L}\sigma_i^z\right)\varrho_j^{x,y} \qquad j = 1\ldots L \\
&\sigma_j^z \to \sigma_j^z \qquad \varrho_j^z \to \varrho_j^z
\end{aligned}
\tag{1.20}
$$

the Hamiltonian (1.18) decouples into a sum of two independent anisotropic XY chains

$$
\begin{aligned}
&H(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\omega_{12},\omega_{34}) \\
&\quad = -\tfrac{1}{2}\sum_{j=1}^{L-1}\big[\omega_{12}(\gamma_1^{-1}\gamma_2\sigma_j^x\sigma_{j+1}^x+\gamma_1\gamma_2^{-1}\sigma_j^y\sigma_{j+1}^y+\gamma_1\gamma_2\sigma_j^z+\gamma_1^{-1}\gamma_2^{-1}\sigma_{j+1}^z) \\
&\qquad +\omega_{34}\,(\gamma_3^{-1}\gamma_4\varrho_j^x\varrho_{j+1}^x+\gamma_3\gamma_4^{-1}\varrho_j^y\varrho_{j+1}^y+\gamma_3\gamma_4\varrho_j^z+\gamma_3^{-1}\gamma_4^{-1}\varrho_{j+1}^z)\big]
\end{aligned}
\tag{1.21}
$$

where ω_{12} and ω_{34} appear as normalization constants.

As an application we finally consider the Hamiltonian (1.18) for a particular choice of the parameters ω_{ij} so that the strength of the couplings between the two XY chains can be controlled by a single parameter ξ. Computing the corresponding spectrum we observe that for $\xi = \pm 1$ the interaction becomes singular so that one obtains 2^{L+1}-fold (instead of four-fold) degenerations. The supplementary symmetry is caused by $L-1$ additional zero modes. Normally, zero modes are known to be exponential modes acting globally on the whole chain. Contrarily the additional zero modes turn out to act only in a specific part of the chain. We thereby find a new type of zero modes which cannot be observed in the case of two-state models.

Another interesting approach towards a generalization of two-state models is to consider supersymmetric quantum chains [16]. Following these ideas it is possible to introduce an integrable supersymmetric generalization of the XY chain. However, such a model always decouples into sectors described by conventional XY models with *site-dependent* coupling constants, and therefore there is no connection to the present type of generalizations (where we have site-independent interactions). In particular if one tries to restore a quantum group symmetry in a supersymmetric XY chain, one always recovers the usual two-dimensional algebra (1.3).

The paper is organized as follows. In section 2 we define the class of quantum chains to be investigated and outline the diagonalization method. In section 3 we derive a general condition for the existence of fermionic zero modes. Section 4 discusses the structure of the corresponding quantum algebra which is a multi-parameter deformation of the Clifford algebra. It is also shown that if some of the deformation parameters coincide, additional algebra automorphisms allow the number of free parameters to be reduced. In section 5 we turn our attention to a particular physical four-state model. We discuss our results and consider a special case where additional zero modes occur. Finally we summarize our conclusions in section 6. In an appendix we show that a recently discovered duality property of the anisotropic XY chain [17] also exists in the generalized case.

2. Multifermionic chains and their diagonalization

In order to define a natural generalization of the XY chain, we first rewrite the fermionic version of the Hamiltonian (1.2) in the general bilinear form

$$
H(\mathbf{A},\mathbf{B},\mathbf{C}) = \tfrac{1}{2}\mathrm{i}\sum_{j=1}^{L-1}\sum_{\mu,\nu=1}^{2n}\left(\mathbf{A}^{\mu,\nu}\tau_j^\mu\tau_{j+1}^\nu+\tfrac{1}{2}\mathbf{B}^{\mu,\nu}\tau_j^\mu\tau_j^\nu+\tfrac{1}{2}\mathbf{C}^{\mu,\nu}\tau_{j+1}^\mu\tau_{j+1}^\nu\right)
\tag{2.1}
$$

where $n=1$ and

$$
\mathbf{A}=\begin{pmatrix}0 & -\eta^{-1}\\ \eta & 0\end{pmatrix}\qquad \mathbf{B}=\begin{pmatrix}0 & q\\ -q & 0\end{pmatrix}\qquad \mathbf{C}=\begin{pmatrix}0 & q^{-1}\\ -q^{-1} & 0\end{pmatrix}.
\tag{2.2}
$$

We consider the Hamiltonian (2.1) for $n > 1$ pairs of local fermionic operators τ_j^μ per site. These operators are supposed to obey the Clifford algebra

$$\{\tau_i^\mu, \tau_j^\nu\} = 2\,\delta_{ij}\delta^{\mu\nu} \qquad i, j = 1\ldots L \quad \mu, \nu = 1, \ldots, 2n \tag{2.3}$$

and therefore the $2n \times 2n$ matrices $\mathbf{B}$ and $\mathbf{C}$ can be assumed to be antisymmetric (since symmetric contributions would result in an irrelevant constant). This defines a $2n$-state model which is a natural generalization of the XY chain in the fermionic language. As we will see below, the four-state Hamiltonian (1.18) just corresponds to the case $n = 2$.

We are now going to diagonalize these generalized quantum chains. Since the Hamiltonian (2.1) is bilinear in the operators τ_j^μ, it is possible to apply standard methods described in [18]. Accordingly H can be written in the diagonal form

$$H(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \sum_{k=0}^{L-1} \sum_{\gamma=1}^{n} \Lambda_k^\gamma \mathrm{i} T_k^{2\gamma} T_k^{2\gamma-1} \tag{2.4}$$

where Λ_k^γ are fermionic excitation energies and T_k^μ are Clifford operators as well

$$\left\{T_k^\mu, T_l^\nu\right\} = 2\,\delta_{kl}\delta^{\mu\nu} \qquad k, l = 0, \ldots, L-1 \quad \mu, \nu = 1, \ldots, 2n\,. \tag{2.5}$$

They are related to the local fermionic operators τ_j^μ by an orthogonal transformation:

$$T_k^\nu = \sum_{j=1}^{L} \sum_{\mu=1}^{2n} \psi_{k,j}^{\nu,\mu}\, \tau_j^\mu \qquad \sum_{j=1}^{L} \sum_{\mu=1}^{2n} \psi_{k,j}^{\gamma,\mu} \psi_{l,j}^{\delta,\mu} = \delta_{kl}\delta^{\gamma\delta}\,. \tag{2.6}$$

Thus the expressions $\mathrm{i}T_k^{2\gamma} T_k^{2\gamma-1}$ in (2.4) commute pairwise and have the eigenvalues ± 1 so that the knowledge of all excitation energies Λ_k^γ allows the spectrum of the Hamiltonian (2.1) to be constructed by taking all combinations into account. As shown in [18] the excitation energies Λ_k^γ and the transformation coefficients $\psi_{k,j}^{\gamma,\mu}$ are solutions of the eigenvalue problem

$$\sum_{j,\nu} M_{ij}^{\mu\nu} \Phi_{kj}^{\gamma\nu} = \mp 2\mathrm{i}\Lambda_k^\gamma\, \Phi_{ki}^{\gamma\mu} \qquad \gamma = 1, \ldots, n \qquad \mu = 1, \ldots, 2n \tag{2.7}$$

where $\Phi_{ki}^{\gamma\mu} = \psi_{k,i}^{2\gamma-1,\mu} \pm \psi_{k,i}^{2\gamma,\mu}$ and $\mathbf{M}$ is the following $2nL \times 2nL$ matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} & \mathbf{A} & & & & \\ -\mathbf{A}^T & \mathbf{B}+\mathbf{C} & \mathbf{A} & & & \\ & -\mathbf{A}^T & \mathbf{B}+\mathbf{C} & \mathbf{A} & & \\ & & \cdots & \cdots & \cdots & \\ & & & -\mathbf{A}^T & \mathbf{B}+\mathbf{C} & \mathbf{A} \\ & & & & -\mathbf{A}^T & \mathbf{C} \end{pmatrix}. \tag{2.8}$$

Since $\mathbf{M}$ is antisymmetric we expect its eigenvalues to occur in pairs with different signs. We thus are free to choose the sign of Λ_k^γ. However, the spectrum does not depend on this choice.

Let us summarize our results at this stage. We have constructed a class of Hamiltonians of the form (2.1) defined on n pairs of fermionic operators per site. These chains depend on $2n^2 - n$ parameters (including one normalization parameter) which are arranged in $2n \times 2n$ matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Their spectra can be determined by solving the reduced eigenvalue problem (2.7). In the following section we are going to derive an additional condition on the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in order to implement the quantum group symmetry and to eliminate non-essential parameters.

3. A condition for the existence of zero modes

In case of the XY chain the quantum group symmetry appears as a fermionic zero mode $\Lambda_0 = 0$ for *arbitrary* parameters η and q. Our aim is to implement a similar structure in the case of generalized chains (2.1). According to (2.7) zero modes are solutions of the system of equations $\sum_{j=1}^{L}\sum_{\nu=1}^{2n} M_{ij}^{\mu\nu}\psi_{0,j}^{\gamma,\nu} = 0$. Since it seems to be impossible to solve this problem in general, one needs an additional condition. In fact, we have shown that the systems of equations simplifies essentially if the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ satisfy the condition

$$\mathbf{A}^T + \mathbf{C}\mathbf{A}^{-1}\mathbf{B} = 0\,. \tag{3.1}$$

This *zero mode condition* is assumed to be valid throughout the rest of this paper. It implies that the components of the zero mode eigenvectors $\psi_{0,j}^{\mu} = (\psi_{0,j}^{\mu,1}, \ldots, \psi_{0,j}^{\mu,2n})$ obey a simple power-law

$$\psi_{0,j}^{\nu} = (-\mathbf{A}^{-1}\mathbf{B})^{j-1}\psi_{0,1}^{\nu}\,. \tag{3.2}$$

It is easy to check that for $n = 1$ the matrices (2.2) satisfy the zero mode condition.

Let us consider the generalized case $n > 1$. In order to simplify the eigenvectors (3.2) and remove unessential parameters one can use the invariance of the Clifford algebra (1.8) under orthogonal transformations $O(2n)$:

$$\tau_i^{\mu} \to \tau_i^{\mu\prime} = \sum_{\nu=1}^{2n} O^{\mu\nu}\tau_i^{\nu} \qquad \mathbf{O}\mathbf{O}^T = \mathbf{O}^T\mathbf{O} = \mathbf{1}\,. \tag{3.3}$$

Therefore a change of basis

$$\mathbf{A} \to \mathbf{A}' = \mathbf{O}\mathbf{A}\mathbf{O}^T \qquad \mathbf{B} \to \mathbf{B}' = \mathbf{O}\mathbf{B}\mathbf{O}^T \qquad \mathbf{C} \to \mathbf{C}' = \mathbf{O}\mathbf{C}\mathbf{O}^T \tag{3.4}$$

corresponds to a similarity transformation of the Hamiltonian (2.1):

$$H(\mathbf{A}', \mathbf{B}', \mathbf{C}') = U\, H(\mathbf{A}, \mathbf{B}, \mathbf{C})U^{-1}\,. \tag{3.5}$$

This allows us to choose a basis where the matrix $-\mathbf{A}^{-1}\mathbf{B}$ in (3.2) is already diagonal:

$$(-\mathbf{A}^{-1}\mathbf{B})^{\mu\nu} = \alpha_\mu\delta^{\mu\nu}\,. \tag{3.6}$$

According to (2.6) the zero mode operators then read

$$T_0^{\mu} = (\alpha_\mu)^{-(L+1)/2}\sum_{j=1}^{L}(\alpha_\mu)^j\tau_j^{\mu} \qquad \mu = 1, \ldots, 2n\,. \tag{3.7}$$

Because of $\Lambda_0^{\gamma} = 0$ (cf equation (2.4)) these operators commute with $H(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and therefore all levels of the spectrum are at least 2^n-fold degenerated. As will be seen in the next section, they appear as the generators of the corresponding quantum algebra.

Another very useful advantage of the zero mode condition (3.1) is a further simplification of the eigenvalue problem (2.7). It turns out that the eigenvalues of $\mathbf{M}$ (beside the zero modes $\Lambda_0^{\mu} = 0$) are the solutions of the polynomial

$$\det\left(-\mathbf{A}^T e^{-i\pi k/L} + (\mathbf{B} + \mathbf{C} - 2i\Lambda_k^{\gamma}) + \mathbf{A}e^{i\pi k/L}\right) = 0 \tag{3.8}$$

where k runs from 1 to $L-1$. This polynomial contains only even powers of Λ_k^{γ} (due to the freedom of choosing its sign) and yields the dispersion relation of the chain.

Notice that the zero modes are always related to exponential wavefunctions and cannot be derived from (3.8). Here it is useful to give some comment. It is a well known property of integrable quantum chains with open boundary conditions that beside excitations with trigonometric wavefunctions there is always a set of exceptional excitations with exponential behaviour. In the thermodynamic limit $L \to \infty$ these wavefunctions are located at the ends *of the chain and have a vanishing energy contribution. In our models a special choice of the* boundary conditions causes these excitation energies to vanish exactly *for finite* L giving the exponential wavefunctions the physical meaning of zero modes.

Hamiltonians of the form (2.1) obeying the zero mode condition (3.1) in the basis (3.6) can be constructed by choosing an arbitrary diagonal $2n \times 2n$ matrix Γ and an arbitrary antisymmetric $2n \times 2n$ matrix Ω so that the matrices

$$\mathbf{A} = -\Gamma\Omega\Gamma^{-1} \qquad \mathbf{B} = \Gamma\Omega\Gamma \qquad \mathbf{C} = \Gamma^{-1}\Omega\Gamma^{-1} \tag{3.9}$$

satisfy the zero mode condition (3.1) and $-\mathbf{A}^{-1}\mathbf{B} = \Gamma^2$ is already diagonal. The corresponding Hamiltonian $H(\Omega, \Gamma)$ therefore depends on $2n^2 + n$ parameters (including one normalization parameter). Let us illustrate this construction for the case $n = 2$. With

$$\Omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \gamma_3 & \\ & & & \gamma_4 \end{pmatrix} \tag{3.10}$$

we obtain a ten-parameter Hamiltonian with two fermionic zero modes. Their deformation parameters α_μ in (3.7) are simply given by $\alpha_\mu = \gamma_\mu^2$. Then by means of a generalized Jordan–Wigner transformation

$$\begin{aligned} \tau_j^1 &= \left(\prod_{i=1}^{j-1} \sigma_i^z \rho_i^z\right) \sigma_j^x & \qquad \tau_j^2 &= \left(\prod_{i=1}^{j-1} \sigma_i^z \rho_i^z\right) \sigma_j^y \\ \tau_j^3 &= \left(\prod_{i=1}^{j-1} \sigma_i^z \rho_i^z\right) \sigma_j^z \rho_j^x & \qquad \tau_j^4 &= \left(\prod_{i=1}^{j-1} \sigma_i^z \rho_i^z\right) \sigma_j^z \rho_j^y \end{aligned} \tag{3.11}$$

one is led directly to the ten-parameter Hamiltonian (1.18). It is now clear that the somewhat artificial appearance of this Hamiltonian is nothing but a simple consequence of Jordan–Wigner factors while in the fermionic formulation the generalization is a quite natural one.

We now apply (3.8) in order to compute the spectrum of the Hamiltonian (2.1). One obtains the fermionic excitation energies

$$\Lambda_k^{1,2} = \sqrt{p_k \pm \sqrt{p_k^2 - q_k}} \qquad k = 1, \ldots, L-1 \tag{3.12}$$

where

$$p_k = \frac{1}{2} \sum_{1 \leqslant \mu < \nu \leqslant 4} \left(\cos\frac{\pi k}{L} - \frac{\alpha_\mu + \alpha_\mu^{-1}}{2}\right)\left(\cos\frac{\pi k}{L} - \frac{\alpha_\nu + \alpha_\nu^{-1}}{2}\right)\omega_{\mu\nu}^2 \tag{3.13}$$

$$q_k = (\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23})^2 \prod_{\mu=1}^{4} \left(\cos\frac{\pi k}{L} - \frac{\alpha_\mu + \alpha_\mu^{-1}}{2}\right). \tag{3.14}$$

The levels of the spectrum can be computed by taking all fermionic combinations into account (see equation (2.4)). Because of the zero modes $\Lambda_0^1 = \Lambda_0^2 = 0$ each level is at least four-fold degenerated. Obviously the spectrum is massless if at least one of the deformation parameters is on the unit circle. Moreover we observe that the spectrum is invariant under discrete transformations $\alpha_\mu \to \alpha_\mu^{-1}$. This symmetry is related to a generalized duality property and will be discussed in the appendix.

4. The Clifford quantum algebra

If the Hamiltonian (2.1) satisfies the zero mode condition (3.1), it is invariant under a $2n$-parameter deformation of the Clifford algebra. This quantum algebra is defined by the commutation relations

$$\{T^\mu, T^\nu\} = 2\delta^{\mu\nu}[E]_{\alpha_\mu} \qquad [E, T^\mu] = 0 \qquad \mu, \nu = 1, \ldots, 2n \tag{4.1}$$

and the coproducts

$$\Delta(T^\mu) = \alpha_\mu^{E/2} \otimes T^\mu + T^\mu \otimes \alpha_\mu^{-E/2} \qquad \mu = 1, \ldots, 2n \tag{4.2}$$

$$\Delta(E) = E \otimes \mathbf{1} + \mathbf{1} \otimes E \qquad \mu, \nu = 1, \ldots, 2n \tag{4.3}$$

with the co-unit $\epsilon(T^\mu) = \epsilon(E) = 0$ and the antipode $S(T^\mu) = T^\mu$ and $S(E) = -E$. This algebra has been given in a similar form in [19], where $2n$ distinct central elements and one deformation parameter have been used (instead of $2n$ deformation parameters and one central element E in our case which leads to a different representation theory). Notice that by construction of our model the dimension of the algebra (4.1) is always even (the odd case, however, is also possible but not of interest in this paper). If all deformation parameters $\alpha_1, \ldots, \alpha_{2n}$ are equal to one, the algebra reduces to the (classical) Clifford algebra $\{T^\mu, T^\nu\} = 2\,\delta^{\mu\nu}$. Beside the trivial one-dimensional representation $T^\mu = E = 0$ the algebra (4.1) possesses only $2n$-dimensional irreducible representations of the form

$$T^\mu = \sqrt{[e]_{\alpha_\mu}}\, t^\mu \qquad E = e\,\mathbf{1} \tag{4.4}$$

where e is a number and the t^μ denote a canonical representation of the $2n$-dimensional classical Clifford algebra $\{t^\mu, t^\nu\} = 2\delta^{\mu\nu}$. For $n = 2$ a possible choice is

$$t^{+1} = \sigma^x \otimes \mathbf{1} \qquad t^{-1} = \sigma^y \otimes \mathbf{1} \qquad t^{+2} = \sigma^z \otimes \sigma^x \qquad t^{-2} = \sigma^z \otimes \sigma^y\,. \tag{4.5}$$

In particular, the fermionic representation corresponds to taking $e = 1$. The coproduct (4.2) then explicitly reads

$$\Delta(t^\mu) = \alpha_\mu^{1/2}\, t^{2n+1} \otimes t^\mu + \alpha_\mu^{-1/2}\, t^\mu \otimes \mathbf{1} \tag{4.6}$$

where $t^{2n+1} = \sigma^z \otimes \sigma^z$ plays the role of a grading operator. By a multiple application of this coproduct we obtain the L-site representation

$$T^\mu = \sum_{j=1}^{L} (\alpha_\mu)^{j-(L+1)/2}\, \tau_j^\mu \qquad E = L \qquad \mu = 1, \ldots, 2n\,. \tag{4.7}$$

These generators are nothing but the zero mode operators defined in (3.7). They commute with the Hamiltonian (2.1) and therefore the chain is invariant under the deformed Clifford algebra (4.1).

If one of the deformation parameters $\alpha_1, \ldots, \alpha_{2n}$ is a root of unity (i.e. the parameters are non-generic) the RHS of (4.1) may vanish. In this case the two-dimensional irreducible representations (4.4) break down and only the trivial one survives. In the spectrum non-generic cases appear as level crossings. Here the Hamiltonian possesses zero-norm eigenvectors and one has to consider an appropriate subspace and a redefined scalar product. However, we do not want to discuss this case and therefore we will assume the deformation parameters to be generic.

If some of the deformation parameters $\alpha_1, \ldots, \alpha_{2n}$ coincide, it is possible to perform orthogonal transformations (3.3) in the corresponding subspace *without* altering the commutation relations (4.1) This allows further parameters in the matrix Ω (see equation (3.10)) to be eliminated. In particular if all deformation parameters coincide, one can always transform the matrix Ω to the block-diagonal form

$$\Omega = \begin{pmatrix} 0 & \omega_{1,2} & & & & \\ -\omega_{1,2} & 0 & & & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & & & & 0 & \omega_{2n-1,2n} \\ & & & & -\omega_{2n-1,2n} & 0 \end{pmatrix}. \tag{4.8}$$

Hence for equal deformation parameters the generalized Hamiltonian (2.1) (together with the zero mode condition (3.1)) always decomposes into a sum of n decoupled isotropic XY chains of the form (1.2). However, the normalizations and deformation parameters q of each copy may be different.

If only $m < 2n$ deformation parameters coincide (while the others are pairwise distinct), similar considerations show that one can remove $\frac{1}{2}(m^2 - m)$ parameters in the matrix Ω by means of $O(m)$ transformations.

5. A special physical example

In this section we want to illustrate our results in the example of the chain (1.18). We only consider a special choice of the coupling constants ω_{ij} to be defined below. This choice is motivated by physical reasons as follows. Thinking of two coupled XY chains and neglecting the influence of the deformation parameters we suppose the internal interactions of each chain to have the same strength ($\omega_{12} = \omega_{34} = 1$). On the other hand there are couplings between both chains which can be controlled by a single parameter ξ. Since within each XY chain only X–X and Y–Y interactions are present, we assume for physical reasons the same to be true for the interactions between both chains, i.e. we exclude X–Y interactions by setting $\omega_{13} = \omega_{24} = 0$. Therefore taking care of the signs in (1.18) a physically reasonable choice of the coupling constant is

$$\omega_{12} = \omega_{34} = 1 \qquad \omega_{14} = -\omega_{23} = \xi \qquad \omega_{13} = \omega_{24} = 0\,. \tag{5.1}$$

The spectrum of the resulting Hamiltonian $H(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \xi)$ is invariant under $\xi \to -\xi$ (to see that, consider the algebra automorphism $\sigma_j^{x,y} \to -\sigma_j^{x,y}$ and $\sigma_j^z \to \sigma_j^z$).

Let us first consider the case where all deformation parameters $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma$ are equal. As shown in figure 1, the spectrum of H depends linearly on ξ in this case. For $\xi = 0$ the spectrum is just the sum of two identical isotropic XY chains

$$H(\gamma, \gamma, \gamma, \gamma, 0) \doteq H^{XY}(1, \gamma^2) \otimes \mathbf{1} + \mathbf{1} \otimes H^{XY}(1, \gamma^2) \tag{5.2}$$

where $H^{XY}(\eta, q)$ is given in (1.2) and '$\doteq$' denotes equality up to similarity transformation. Looking at figure 1 it is obvious that for arbitrary ξ only the normalizations of the two XY chains vary linearly:

$$H(\gamma, \gamma, \gamma, \gamma, \xi) \doteq (1+\xi) H^{XY}(1, \gamma^2) \otimes \mathbf{1} + (1-\xi) \mathbf{1} \otimes H^{XY}(1, \gamma^2)\,. \tag{5.3}$$

Thus for equal deformation parameters the generalized chain (5.3) always decouples into a sum of two isotropic XY chains in agreement with the observation in (4.8). In particular for $\xi = \pm 1$ only one of them survives so that each level of the spectrum is at least 2^{L+1}-fold degenerated. Here the Hamiltonian (5.3) is invariant under local rotations generated by the anticommuting operators

$$S_j^1 = \tau_j^1 - \tau_j^3 \qquad S_j^2 = \tau_j^2 - \tau_j^4 \qquad j = 1, \ldots, L\,. \tag{5.4}$$

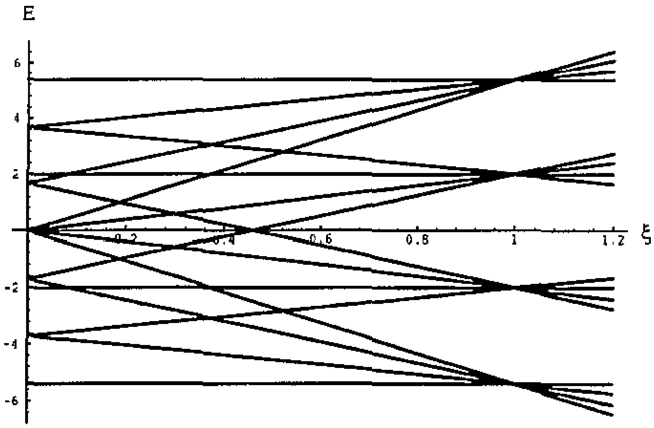


Figure 1. Spectrum of $H(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \xi)$ for three sites.

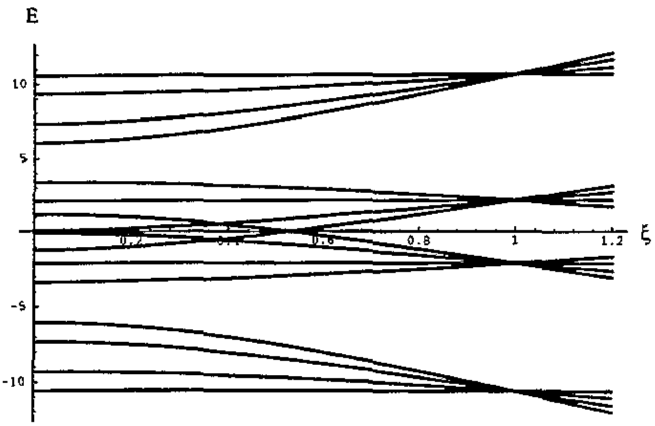


Figure 2. Spectrum of $H(\frac{2}{7}, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}, \xi)$ for three sites.

We will come back to this case below.

For arbitrary deformation parameters the spectrum cannot be decomposed into a sum of XY chain spectra (see figure 2). The only exceptions are $\xi = 0$ and $\xi = \pm 1$. In the first case we have

$$H(\gamma_1, \gamma_2, \gamma_3, \gamma_4, 0) \doteq H^{XY}(\gamma_1^{-1}\gamma_2, \gamma_1\gamma_2) \otimes \mathbf{1} + \mathbf{1} \otimes H^{XY}(\gamma_3^{-1}\gamma_4, \gamma_3\gamma_4) \tag{5.5}$$

in agreement with (1.21). For $\xi = \pm 1$ the spectrum of H coincides (up to degenerations) with the spectrum of a single anisotropic XY chain:

$$H(\gamma_1, \gamma_2, \gamma_3, \gamma_4, 1) \doteq 2H^{XY}(\eta, q) \otimes \mathbf{1}. \tag{5.6}$$

Here η and q are solutions of the trigonometric equations

$$\nu = \left(\frac{\eta + \eta^{-1}}{2}\right)^2 + \left(\frac{q + q^{-1}}{2}\right)^2 - 1 = \tfrac{1}{4}(\Delta_1 + \Delta_3)(\Delta_2 + \Delta_4) \tag{5.7}$$

$$\mu = 4\left(\frac{\eta - \eta^{-1}}{2}\right)^2 \left(\frac{q - q^{-1}}{2}\right)^2 = \tfrac{1}{4}(\Delta_1 + \Delta_3 - \Delta_2 - \Delta_4)^2 \tag{5.8}$$

where $\Delta_i = \frac{1}{2}(\alpha_i + \alpha_i^{-1}) = \frac{1}{2}(\gamma_i^2 + \gamma_i^{-2})$. In order to prove (5.6) we checked that the Hamiltonian $H(\gamma_1, \gamma_2, \gamma_3, \gamma_4, 1) = 2\sum_{j=1}^{L-1} h_j$ satisfies the algebraic relations of the anisotropic XY chain [6]

$$\left[h_j h_{j\pm1} h_j - h_{j\pm1} h_j h_{j\pm1} + (\nu - 1)(h_j - h_{j\pm1})\right](h_j - h_{j\pm1}) = \mu \tag{5.9}$$

$$h_j^2 = \nu\,. \tag{5.10}$$

Hence for arbitrary deformation parameters a special tuning of the coupling constants (due to the choice $\xi = \pm1$) provides a strong symmetry. The spectrum is equivalent to that of a single anisotropic XY chain and each level is at least 2^{L+1}-fold degenerated. The corresponding symmetry operators read

$$\begin{aligned} L_j^1 &= \sum_{k=1}^{j} \left(\alpha_1^{k-j-1/2}\tau_j^1 - \alpha_3^{k-j-1/2}\tau_j^3\right) \qquad j = 1, \ldots, L-1 \\ L_j^2 &= \sum_{k=1}^{j} \left(\alpha_2^{k-j-1/2}\tau_j^2 - \alpha_4^{k-j-1/2}\tau_j^4\right) \\ [L_i^\mu, H] &= 0 \end{aligned} \tag{5.11}$$

and may be understood as $L-1$ additional zero modes. Together with the four zero mode generators T_0^μ they cause 2^{L+1}-fold degenerations of each level.

The most important property of the zero modes (5.11) is that they act only in a part of the chain extending from the left boundary to a certain position j. Similarly, there are zero mode operators acting from position $j+1$ to the right boundary

$$\begin{aligned} R_j^1 &= \sum_{k=j+1}^{L} \left(\alpha_1^{k-j-1/2}\tau_j^1 - \alpha_3^{k-j-1/2}\tau_j^3\right) \qquad j = 1, \ldots, L-1 \\ R_j^2 &= \sum_{k=j+1}^{L} \left(\alpha_2^{k-j-1/2}\tau_j^2 - \alpha_4^{k-j-1/2}\tau_j^4\right). \end{aligned} \tag{5.12}$$

Because of $L_j^\mu + R_j^\mu = \alpha_\mu^{L/2-j} T_0^\mu - \alpha_{\mu+2}^{L/2-j} T_0^{\mu+2}$ only one set of operators (e.g. $\{L_j^\mu\}$) is independent. It is easy to check that for all deformation parameters being equal one retrieves the local symmetry operators (5.4) by taking appropriate linear combinations.

As already mentioned in section 3, the existence of exponential modes is a well known property of integrable chains with non-periodic boundary conditions. Normally there are only four exponential modes in our model, namely the zero modes (3.7). These modes act globally. Contrarily the additional zero modes (5.11) and (5.12) act only to the left and to the right of a certain position, respectively. To our knowledge this phenomenon has not been observed before. It has its origin in a singularity of the interaction ($\det(\Omega) = 0$) for $\xi = \pm1$. Roughly speaking this singularity of the interaction allows certain modifications of the states at site j which do not affect the situation at site $j+1$. Therefore if one combines exponential modes in a appropriate way they 'trickle away' at a certain position.

6. Conclusions

The present work is based on previous investigations of the anisotropic XY chain in a magnetic field with a special kind of boundary conditions. These boundary conditions imply the existence of a fermionic zero mode which is related to a quantum group symmetry.

In this article we found a class of integrable quantum chains which can be understood as generalizations of the XY chain. These $2n$-state models are defined on n fermionic degrees of freedom per site and can be diagonalized in terms of free fermions as well. In analogy to the XY chain case we found a general condition for the existence of fermionic zero modes. This condition in turn implies that the Hamiltonian is invariant under a $2n$-parameter deformation of the $2n$-dimensional Clifford algebra and causes 2^n-fold degenerations of each energy level. Discussing the structure of this algebra we observed that if some of the deformation parameters coincide, the symmetry of the chain is increased by means of orthogonal algebra automorphisms leading to higher degenerations of the spectrum.

The structure of the quantum group allows complicated internal couplings to be implemented. These couplings are non-trivial in the sense that the spectrum of such a chain cannot be decomposed into a sum of XY chain spectra. As an example we discussed a four-state model and computed the corresponding spectrum. In this case one can think of two XY chains with nearest-neighbour couplings between them. The corresponding Hamiltonian depends on ten parameters, four of them being deformation parameters of the Clifford algebra. We restricted our attention to a special choice of the other six parameters which is supposedly the most physical one (we allow only XX and YY couplings between the chains) and illustrated our results. For a special tuning of the coupling constants the interaction matrix becomes singular and the spectrum coincides with that of a single XY chain. However, the degenerations are much larger due to the existence of $L-1$ additional zero modes. In contrast to usual exponential modes, zero modes of this kind act only in a particular part of the chain extending from the left boundary to a certain position.

Acknowledgments

I would like to thank V Rittenberg and I Peschel for many valuable discussions.

Appendix. Discrete symmetry transformations

Beside the quantum group invariance the anisotropic XY chain (1.2) possesses a further important symmetry. Diagonalizing the Hamiltonian (1.2) one observes that the exchange of the parameters η and q does not modify the spectrum. In [17] we derived the corresponding similarity transformation

$$H^{XY}(\eta, q) = U\, H^{XY}(q, \eta)\, U^{-1} \tag{A1}$$

and showed that U reduces in a special limit to the Ising duality transformation. For this reason we denoted the transformation (A1) as 'generalized duality transformation' (although the XY chain is not self-dual in the usual sense). In this section we show that a similar symmetry exists in the case of generalized chains of the form (2.1) obeying the zero mode condition (3.1).

We first notice that in the XY chain case the transformation (A1) just inverts the deformation parameter $\alpha_1 \leftrightarrow \alpha_1^{-1}$ while α_2 is not changed (in the same way it is possible to construct a similarity transformation which inverts α_2 and keeps α_1 fixed). Then looking at the fermionic excitation energies (3.12) of the generalized Hamiltonian (1.18) with two fermionic degrees of freedom per site we recognize that the inversion of any deformation parameter $\alpha_\mu \leftrightarrow \alpha_\mu^{-1}$ ($\mu = 1, \ldots, 4$) does not alter the spectrum. We therefore expect this observation to hold for arbitrary n, i.e. using the notation of (3.9) we assume that for every $\mu = 1, \ldots, 2n$ we have

$$H(\alpha_1, \ldots, \alpha_\mu^{-1}, \ldots, \alpha_{2n}, \Omega) \doteq H(\alpha_1, \ldots, \alpha_\mu, \ldots, \alpha_{2n}, \Omega)\,. \tag{A2}$$

In analogy to the results of [17] it turns out that the corresponding similarity transformation depends exclusively on the deformation parameter it is inverting:

$$H(\alpha_1, \ldots, \alpha_\mu^{-1}, \ldots, \alpha_{2n}, \Omega) = U(\alpha_\mu) H(\alpha_1, \ldots, \alpha_\mu, \ldots, \alpha_{2n}, \Omega) U^{-1}(\alpha_\mu) . \tag{A3}$$

Denoting

$$N_\mu = 2^{L-1} \frac{1 + \alpha_\mu^L}{(1 + \alpha_\mu)^L} \qquad \omega_\mu = \frac{\alpha_\mu^{1/2} - \alpha_\mu^{-1/2}}{\alpha_\mu^{1/2} + \alpha_\mu^{-1/2}} \tag{A4}$$

this transformation can be written in terms of a 'time-ordered' exponential

$$U(\alpha_\mu) = \frac{1}{\sqrt{N_\mu}} T \exp(\omega_\mu G_\mu) \tag{A5}$$

where G_μ is a non-local generator

$$G_\mu = \sum_{1 \leqslant j_1 < j_2 \leqslant L} \tau_{j_1}^\mu \tau_{j_2}^\mu . \tag{A6}$$

T is an ordering operator defined by

$$T \tau_i^\mu \tau_j^\mu = \begin{cases} \tau_i^\mu \tau_j^\mu & i < j \\ -\tau_j^\mu \tau_i^\mu & i > j \\ 0 & i = j . \end{cases} \tag{A7}$$

Explicitly the transformation $U(\alpha_\mu)$ is given by the polynomial

$$U(\alpha_\mu) = \frac{1}{\sqrt{N_\mu}} \left(1 + \sum_{k=0}^{[L/2]} \omega_\mu^k \sum_{1 \leqslant j_1 < j_2 < \ldots < j_{2k} \leqslant L} \tau_{j_1}^\mu \tau_{j_2}^\mu \ldots \tau_{j_{2k}}^\mu \right) \tag{A8}$$

where $[L/2]$ denotes the truncation of $L/2$ to an integer number. It is an orthogonal transformation and its inverse is given by

$$U(\alpha_\mu)^{-1} = U(\alpha_\mu)^T = U(\alpha_\mu^{-1}) . \tag{A9}$$

Therefore $U(\alpha_\mu)$ reduces to the identity if the deformation parameter in question is equal to one. Because of $[G_\mu, G_\nu] = 0$ the transformations $U(\alpha_\mu)$ commute for different μ and can be combined freely. Notice that for non-generic deformation parameters ($\alpha_\mu^L = \pm 1$) the transformation $U(\alpha_\mu)$ does not exist since the normalization N_μ diverges. For that reason the transformation $\prod_{\mu=1}^{2n} U(\alpha_\mu)$ must not be confused with the action of the R matrix of the quantum group which inverts the deformation parameters as well.

It is well known in the theory of quantum groups that the inversion of a deformation parameter corresponds to an algebra homomorphism. The transformation (A5) shows that the same is true for the whole physical system. It should be emphasized that it relates different physical situations (e.g. disordered and frozen states). If the deformation parameter in question is equal to one, the system undergoes a massless phase transition. Two types of transitions are possible. If the dispersion of the massless excitations is linear in k, we have a criticial Ising transition, otherwise if the dispersion is quadratic in k, we observe a Pokrovsky–Talapov phase transition.

References

[1] den Nijs N 1988 *Phase Transitions and Critical Phenomena* vol 12 ed C Domb and J L Lebowitz (New York: Academic Press) p 219

[2] Barouch E, McCoy B M and Dresden M 1970 *Phys. Rev.* A **2** 1075
Barouch E and McCoy B M 1971 *Phys. Rev.* A **3** 786

Vaidya H G and Tracy C A 1978 *Physica* **92A** 1
[3] Siggia E D 1977 *Phys. Rev.* B **16** 2319
[4] Alcaraz F C, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys., NY* **230** 250
[5] Saleur H 1989 *Trieste Conf. on Recent Developements in Conformal Field Theories* ed Randjikar-Daemi, E Sezgin and J B Zuber (Singapore: World Scientific) 160
[6] Hinrichsen H and Rittenberg V 1992 *Phys. Lett.* **275B** 350; 1993 *Phys. Lett.* **304B** 115
[7] Cuerno R and Gonzáles-Ruiz A 1993 *J. Phys. A: Math. Gen.* **26** L605
[8] Schultz C L 1981 *Phys. Rev. Lett.* **46** 629
Perk J H H and Schultz C L 1981 *Non-Linear Integrable Systems, Classical Theory and Quantum Theory* ed M Jimbo and T Miwa (Singapore: World Scientific)
[9] Martin P P and Rittenberg V 1992 *Int. J. Mod. Phys.* B **4** 797
[10] Batchelor M T, Mezincescu L, Nepomechie R I and Rittenberg V 1990 *J. Phys. A: Math. Gen.* **23** L141
Meljanac S, Mileković M and Pallua S 1990 *J. Phys. A: Math. Gen.* **24** 581
[11] Jimbo M, Miki K, Miwa T and Nakayashiki A 1992 *Phys. Lett.* **168A** 256
Davies B, Foda O, Jimbo M, Miwo T and Nakayashiki A 1993 *Commun. Math. Phys.* **151** 89
[12] Alcaraz F C, Grimm U and Rittenberg V 1989 *Nucl. Phys.* B **316** 735
Grimm U and Schütz G 1993 *J. Stat. Phys.* **71** 923
[13] Karowski M and Zapletal A 1994 *Preprint* SFB-288-97 (*Nucl. Phys.* B to appear)
[14] Suzuki M 1971 *Prog. Theor. Phys.* **46** 1337
[15] López E 1994 *J. Phys. A: Math. Gen.* **27** 845
[16] Baker T H and Jarvis P D Quantum superspin chains *Preprint* UTAS-PHYS-94-10 (*Int. J. Mod. Phys.* submitted)
[17] Hinrichsen H 1994 *J. Phys. A: Math. Gen.* **27** 1121
[18] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys., NY* **16** 407
[19] Cuerno R, Gómez C, López E and Sierra G 1993 *Phys. Lett.* **307B** 56